

Hamiltonians for a general dilaton gravity theory on a spacetime with a non-orthogonal, timelike or spacelike outer boundary

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Abstract

A generalization of two recently proposed general relativity Hamiltonians, to the case of a general $(d+1)$ -dimensional dilaton gravity theory in a manifold with a timelike or spacelike outer boundary, is presented.

1 Introduction

The study of Hamiltonians for general relativity and other gravity theories is important for many interrelated questions and issues, such as black hole thermodynamics, in particular black hole entropy and its statistical origin, or as the definition of quasilocal quantities. In particular, the boundary terms, which are part of the Hamiltonian, are especially relevant. In fact, the Hamiltonian reduces to them when evaluated on-shell and they are used to determine global charges and thermodynamic quantities.

The form of the Hamiltonian boundary terms depends on the boundary conditions we use for the variational principle (for instance we can choose to fix the metric induced on the boundary), or on gauge conditions such as, for instance, the orthogonality of the boundaries. In the general framework of the Arnowitt-Deser-Misner parametrization, three different gravitational Hamiltonians have

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been proposed recently by, respectively, Hawking and Hunter (HH) [1], Booth and Mann (BM) [2], Creighton and Mann (CM) [3].

The HH and BM Hamiltonians correspond to different Legendre transformations of the Einstein-Hilbert action and represent the natural choice for classical general relativity defined on a spacetime manifold with non-orthogonal boundaries. Conversely, the CM proposal gives the Hamiltonian for a dilaton gravity theory defined on a spacetime manifold with orthogonal boundaries.

The limitations of the different proposal are evident. If one wants to describe dilaton gravity theories in the Hamiltonian framework, one has to use the CM prescription and is therefore forced to consider only orthogonal boundaries. In some situation this limitation may be too strong, for instance if one wants to consider symmetry transformations whose generators cannot be tangent to a timelike boundary (e.g. spatial or null translations). This kind of generators are important in the discussion of the asymptotic symmetries of the spacetime and the associated charges [4, 5, 6, 7]. On the other hand, if one needs to consider a manifold with non-orthogonal boundaries, one can use the HH or BM prescription but is limited to the non-dilatonic case.

Therefore, it is natural to investigate the possibility of using a Booth-Mann-like Hamiltonian together with *any* evolution generator. In this paper we show that this is possible. We propose two Hamiltonians for a general dilaton gravity theory defined on a $(d+1)$ -dimensional spacetime with non-orthogonal boundaries. Our Hamiltonians generalize and comprehend the HH, BM, and CM Hamiltonians. Moreover, they can deal with spacelike, as well as timelike, outer boundaries.

The structure of the paper is the following. In Sect. 2 we set up our notation and define the objects we are dealing with. In Sect. 3 we derive our Hamiltonians. In Sect. 4 we discuss our results.

2 Definitions

We consider a $(d+1)$ -dimensional spacetime manifold \mathcal{M} whose boundary consists of two spacelike hypersurfaces \mathcal{S}' and \mathcal{S}'' sharing the same topology, and an ‘outer’ hypersurface \mathcal{B} , which can be either timelike or spacelike, with topology $\partial\mathcal{S}' \times \mathcal{I}$, where \mathcal{I} is a real interval. The spacetime is foliated into spacelike hypersurfaces \mathcal{S}_t of constant t , where $t : \mathcal{M} \rightarrow \mathbb{R}$ is a time function defined throughout \mathcal{M} . The initial and final hypersurfaces of this foliation are \mathcal{S}' and \mathcal{S}'' . Another foliation is induced on the boundary \mathcal{B} and is given by spacelike surfaces $\mathcal{P}_t = \mathcal{S}_t \cap \mathcal{B}$ of dimension $(d-1)$. The initial and final surfaces are $\mathcal{P}' = \mathcal{S}' \cap \mathcal{B}$ and $\mathcal{P}'' = \mathcal{S}'' \cap \mathcal{B}$, respectively. We can also think of every \mathcal{P}_t as given by the intersection of \mathcal{B} with (local) orthogonal hypersurfaces $\tilde{\mathcal{S}}_t$.¹

The metric on the spacetime \mathcal{M} is $g_{\mu\nu}$, with volume element $\sqrt{-g}$, covariant

¹When the \mathcal{B} boundary is spacelike, we only assume locality and do not suppose that the hypersurfaces $\tilde{\mathcal{S}}_t$ foliate the whole spacetime \mathcal{M} . In this case $\tilde{\mathcal{S}}_t$ is timelike and we do not want a foliation of \mathcal{M} into timelike hypersurfaces. Thus, *clamped foliations*, in the sense of Lau [8, 9], are allowed only when \mathcal{B} is timelike.

derivative ∇_μ and curvature $R_{\mathcal{M}}$. With respect to this metric the \mathcal{M} -foliation lapse is $N \stackrel{\text{def}}{=} [-(\nabla t)^2]^{-1/2}$. The metric $g_{\mu\nu}$ induces other metric structures on the various surfaces. These structures are described in detail below.

A scalar dilaton field η is also defined on \mathcal{M} , as well as its functions $f : \eta \mapsto f(\eta)$, $k : \eta \mapsto k(\eta)$ and $p : \eta \mapsto p(\eta)$; their derivatives $\frac{df}{d\eta}$ etc. are written as f' etc.; their restrictions to the various surfaces, $\eta|_{\mathcal{S}_t}$, $\eta|_{\mathcal{B}}$, $f|_{\mathcal{P}_t}$, etc., will be often called η , f , etc. for simplicity.

The Lie derivative operator is denoted by \mathbf{L} .

The \mathcal{S}_t hypersurfaces

A future-pointing vector field u^μ normal to every \mathcal{S}_t is defined on \mathcal{M} ; its acceleration $a^\mu \stackrel{\text{def}}{=} u^\nu \nabla_\nu u^\mu$ is tangent to \mathcal{S}_t ($u_\mu a^\mu = 0$). The induced Riemannian metric on every \mathcal{S}_t is $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, with volume element \sqrt{h} (note that $\sqrt{h}N = \sqrt{-g}$), covariant derivative \mathbf{D}_μ , intrinsic curvature $R_{\mathcal{S}}$, and extrinsic curvature $K_{\mu\nu} \stackrel{\text{def}}{=} -h_\mu^\tau \nabla_\tau u_\nu$. Tensors are projected onto the \mathcal{S}_t hypersurfaces by $h^\mu{}_\nu$.

The \mathcal{B} boundary

The outer boundary \mathcal{B} , whose normal \tilde{n}^μ we require to be always outward-pointing, can be timelike (see Fig. 1) or spacelike. When \mathcal{B} is spacelike we have to distinguish between two different cases, sketched in Figs. 2 and 3. In Fig. 2 \mathcal{B} lies outside the future of \mathcal{S}' (its normal is past-pointing), while it lies inside the future of \mathcal{S}' in Fig. 3 (its normal is future-pointing). We can characterize the three different cases defining the following quantities:

$$\varepsilon \stackrel{\text{def}}{=} \tilde{n}_\mu \tilde{n}^\mu, \quad (1)$$

$$\beta \stackrel{\text{def}}{=} u^\mu \tilde{n}_\mu, \quad (2)$$

as well as the hyperbolic angle α :

$$\alpha \stackrel{\text{def}}{=} \begin{cases} -\text{sgn}(\beta) \text{arccosh} |\beta| & \text{if } \varepsilon = -1, \\ \text{arcsinh} \beta & \text{if } \varepsilon = +1. \end{cases} \quad (3)$$

If \mathcal{B} is timelike (see Fig. 1) then $\varepsilon = +1$, $\beta \gtrless 0$, and $\alpha \gtrless 0$; if \mathcal{B} is spacelike and outside the future of \mathcal{S}' (past-pointing \tilde{n}^μ , see Fig. 2) then $\varepsilon = -1$, $\beta \geq 1$, and $\alpha \leq 0$; finally, if \mathcal{B} is spacelike and inside the future of \mathcal{S}' (future-pointing \tilde{n}^μ , see Fig. 3) then $\varepsilon = -1$, $\beta \leq 1$, and $\alpha \geq 0$. Notice that we cannot pass smoothly from one spacelike case to the other.

On \mathcal{B} we have the induced intrinsic metric $\gamma_{\mu\nu} = g_{\mu\nu} - \varepsilon \tilde{n}_\mu \tilde{n}_\nu$ with volume element $\sqrt{|\gamma|}$, covariant derivative Δ_μ and extrinsic curvature $\Theta_{\mu\nu} \stackrel{\text{def}}{=} -\gamma_\mu^\tau \nabla_\tau \tilde{n}_\nu$. In the $\varepsilon = +1$ case the induced metric is Lorentzian with signature $(-++\dots)$, while in the $\varepsilon = -1$ case it is a positive definite Riemannian metric having signature $(++++\dots)$. We can project tensors onto \mathcal{B} by using $\gamma^\mu{}_\nu$.

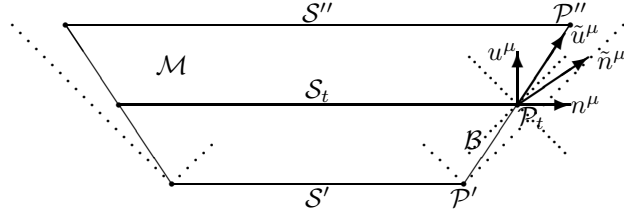


Figure 1: Example of foliation of a two-dimensional spacetime \mathcal{M} with a timelike outer boundary ($\varepsilon=+1$, $\beta \gtrless 0$, $\alpha \gtrless 0$). Dotted lines represent lightcones.

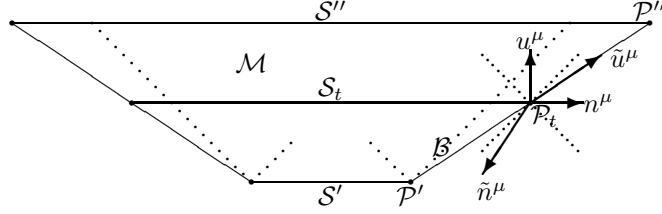


Figure 2: Example of foliation of a two-dimensional spacetime \mathcal{M} with a spacelike outer boundary outside the future of S' ($\varepsilon=-1$, $\beta \geq +1$, $\alpha \leq 0$). Dotted lines represent lightcones.

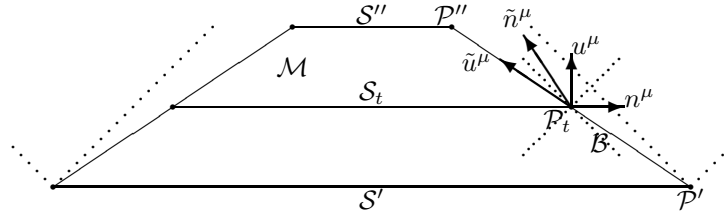


Figure 3: Example of foliation of a two-dimensional spacetime \mathcal{M} with a spacelike outer boundary inside the future of S' ($\varepsilon=-1$, $\beta \leq -1$, $\alpha \geq 0$). Dotted lines represent lightcones.

The spacetime foliation induces a foliation in \mathcal{B} by means of the induced time function $t|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathbb{R}$, and the associated lapse is $\tilde{N} \stackrel{\text{def}}{=} [-\varepsilon(\Delta t|_{\mathcal{B}})^2]^{-1/2}$.

The \mathcal{P}_t surfaces

The \mathcal{P}_t surfaces are defined by the intersection of the outer boundary with the various slices \mathcal{S}_t , so they can be viewed as embedded in \mathcal{B} or in \mathcal{S}_t . In particular, \mathcal{P}' and \mathcal{P}'' together form the boundary of \mathcal{B} , and every \mathcal{P}_t is the boundary of \mathcal{S}_t . Hence, four different unit normal vector fields can be defined on \mathcal{P}_t : as a surface in \mathcal{S}_t , \mathcal{P}_t has the outward-pointing spacelike normal n^μ , and shares with \mathcal{S}_t the future-pointing timelike unit normal u^μ ; as a surface in \mathcal{B} , \mathcal{P}_t has the ‘future-pointing’ unit normal \tilde{u}^μ , and shares with \mathcal{B} the outward-pointing unit normal \tilde{n}^μ . Both n^μ and \tilde{u} can be obtained by projection of the normals \tilde{n}^μ and u^μ onto \mathcal{S}_t and onto \mathcal{B} respectively, and normalizing,

$$n^\mu = \varepsilon \lambda h^\mu{}_\nu \tilde{n}^\mu = \varepsilon \lambda \tilde{n}^\mu + \varepsilon \lambda \beta u^\mu, \quad (4a)$$

$$\tilde{u}^\mu = \varepsilon \lambda \gamma^\mu{}_\nu u^\mu = \frac{1}{\lambda} u^\mu - \lambda \beta \tilde{n}^\mu, \quad (4b)$$

where the normalizing positive scalar λ is defined by:

$$\lambda = \frac{1}{\sqrt{\varepsilon + \beta^2}} = \begin{cases} (\sinh |\alpha|)^{-1} & \text{if } \varepsilon = -1, \\ (\cosh \alpha)^{-1} & \text{if } \varepsilon = +1 \end{cases} \quad (5)$$

(we note that $\delta\beta = \varepsilon \frac{1}{\lambda} \delta\alpha$). Notice that $n_\mu n^\mu = +1$, $u_\mu u^\mu = -1$, $\tilde{u}_\mu \tilde{u}^\mu = -\varepsilon$, $\tilde{n}_\mu \tilde{n}^\mu = \varepsilon$, and that the following relations hold:

$$\tilde{n}^\mu = \varepsilon \frac{1}{\lambda} n^\mu - \beta u^\mu \quad \tilde{u}^\mu = \frac{1}{\lambda} u^\mu - \varepsilon \beta n^\mu, \quad (6a)$$

$$n^\mu = \frac{1}{\lambda} \tilde{n}^\mu + \beta \tilde{u}^\mu \quad u^\mu = \varepsilon \frac{1}{\lambda} \tilde{u}^\mu + \varepsilon \beta \tilde{n}^\mu. \quad (6b)$$

The Riemannian metric induced on \mathcal{P}_t is

$$\sigma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu + u_\mu u_\nu = g_{\mu\nu} + \varepsilon \tilde{u}_\mu \tilde{u}_\nu - \varepsilon \tilde{n}_\mu \tilde{n}_\nu, \quad (7)$$

with volume element $\sqrt{\sigma}$ (and $\sqrt{\sigma} \tilde{N} = \sqrt{|\gamma|}$). Tensors are projected on \mathcal{P}_t by using $\sigma^\mu{}_\nu$. On \mathcal{P}_t we can define two extrinsic curvatures $\theta_{\mu\nu}$, $\tilde{\theta}_{\mu\nu}$,

$$\theta_{\mu\nu} \stackrel{\text{def}}{=} -\sigma_\mu^\tau \sigma_\nu^\sigma \nabla_\tau n_\sigma = -\sigma_\mu^\tau \mathbf{D}_\tau n_\nu, \quad (8)$$

is defined with respect to the embedding in \mathcal{S}_t , whereas

$$\tilde{\theta}_{\mu\nu} \stackrel{\text{def}}{=} -\sigma_\mu^\tau \sigma_\nu^\sigma \nabla_\tau \tilde{n}_\sigma, \quad (9)$$

is defined with respect to the embedding in a hypersurface $\tilde{\mathcal{S}}_t$, locally orthogonal to \mathcal{B} (so that $\tilde{\mathcal{S}}_t$ has unit normal \tilde{u}^μ). The following useful relation holds among the traces of the extrinsic curvatures defined so far:

$$\mathbf{tr} \theta = \varepsilon \lambda \mathbf{tr} \Theta + \varepsilon \lambda \beta \mathbf{tr} K + \varepsilon \lambda \tilde{n}^\mu a_\mu - \lambda \tilde{u}^\mu \nabla_\mu \alpha, \quad (10)$$

$$= \varepsilon \mathbf{tr} \tilde{\theta} + \varepsilon \lambda \beta \mathbf{tr} K + \varepsilon \lambda \beta n^\mu \tilde{n}^\nu \nabla_\mu \tilde{u}_\nu + \varepsilon \lambda \beta n^\mu \nabla_\mu \alpha. \quad (11)$$

Bulk and boundary foliations

The time evolution of the hypersurfaces \mathcal{S}_t (and of the fields defined on them) can be specified by means of a time-flow vector field X^μ , satisfying $\mathbf{d}t(X) \equiv 1$. An equivalent definition is,

$$X^\mu = Nu^\mu + V^\mu, \quad (12)$$

where $N \equiv [-(\nabla t)^2]^{-1/2} = -u_\mu X^\mu$ and $V^\mu \equiv h^\mu{}_\nu X^\nu$ are the (bulk) lapse and shift, respectively.

Analogously, the time evolution of the surfaces \mathcal{P}_t along \mathcal{B} can be specified by a boundary time-flow vector field \tilde{X}^μ tangent to \mathcal{B} and such that $\mathbf{d}t|_{\mathcal{B}}(\tilde{X}) \equiv 1$. We have now,

$$\tilde{X}^\mu = \tilde{N}\tilde{u}^\mu + \tilde{V}^\mu, \quad (13)$$

where $\tilde{N} \equiv [-\varepsilon(\Delta t|_{\mathcal{B}})^2]^{-1/2} = -\varepsilon\tilde{u}_\mu\tilde{X}^\mu$ and $\tilde{V}^\mu \equiv \gamma^\mu{}_\nu\tilde{X}^\nu$ are the *boundary* lapse and shift.

In general, the bulk time-flow vector field X^μ is not tangent to the outer boundary, $\tilde{n}_\mu X^\mu \neq 0$, so that it differs from the boundary time-flow vector field: $X^\mu|_{\mathcal{B}} \neq \tilde{X}^\mu$. This means that the bulk and boundary shifts are unrelated to each other. Note, though, that the bulk and boundary lapses N and \tilde{N} are always related by:

$$\tilde{N} = \lambda N. \quad (14)$$

This equation is just a consequence of the fact that the \mathcal{B} foliation is induced by the \mathcal{M} foliation.

When the vector field X^μ is tangent to \mathcal{B} , $\tilde{n}_\mu X^\mu = 0$, we may require the two time-flow vector fields to coincide, $X^\mu|_{\mathcal{B}} = \tilde{X}^\mu$, so that the respective shifts are related by

$$\tilde{V}^\mu = \sigma^\mu{}_\nu V^\nu = V^\mu + \varepsilon\tilde{N}\beta n^\mu. \quad (15)$$

The general action

The action for a general dilaton gravity theory on a $(d+1)$ -dimensional spacetime is [3, 10]

$$\begin{aligned} \mathcal{J} \stackrel{\text{def}}{=} & \frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-g} [f(\eta) R_{\mathcal{M}} + k(\eta) (\nabla\eta)^2 + p(\eta)] \\ & + \frac{1}{\kappa} \int_{\mathcal{S}'}^{\mathcal{S}''} \sqrt{h} f(\eta) \mathbf{tr} K - \frac{\varepsilon}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} f(\eta) \mathbf{tr} \Theta + \frac{1}{\kappa} \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\sigma} f(\eta) \alpha, \end{aligned} \quad (16)$$

where the surface terms are chosen in such way that the metric and dilaton field induced on the boundary can be held fixed when the action is varied.

In this equation, f , k , p , and the constant κ depend on the model under consideration (for example, setting $d = 1$, $\kappa = \pi$, $\eta = e^{-2\phi}$, $f(\eta) = \eta$, $k(\eta) = 0$,

$p(\eta) = -2\Lambda\eta$, we have the Jackiw-Teitelboim action [11]; setting $d = 3$, $\kappa = 8\pi G$, $\eta = f \equiv 1$, $k = p \equiv 0$, we have the classical Einstein-Hilbert action; cf. Lemos [12]). We have not included minimally-coupled matter terms in the action, because the presence of these terms does not affect our main results.

The variation of the action (16) can be computed using a generalization of Kijowski's calculation [13],

$$\begin{aligned} \delta\mathcal{I} = & \int_{\mathcal{M}} (\Xi^{\mu\nu} \delta g_{\mu\nu} + \Xi \delta\eta) + \int_{\mathcal{S}'}^{S''} (\mathbf{P}^{\mu\nu} \delta h_{\mu\nu} + \mathbf{P} \delta\eta) \\ & + \int_{\mathcal{B}} (\mathbf{\Pi}^{\mu\nu} \delta \gamma_{\mu\nu} + \mathbf{\Pi} \delta\eta) + \int_{\mathcal{P}'}^{\mathcal{P}''} (\boldsymbol{\pi}^{\mu\nu} \delta \sigma_{\mu\nu} + \boldsymbol{\pi} \delta\eta), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \Xi^{\mu\nu} = & -\frac{1}{2\kappa} \sqrt{-g} [f R_{\mathcal{M}}^{\mu\nu} - \frac{1}{2} R_{\mathcal{M}} g^{\mu\nu} - \frac{1}{2} p g^{\mu\nu} + g^{\mu\nu} \nabla^2 f \\ & - \nabla^\mu \nabla^\nu f + k \nabla^\mu \eta \nabla^\nu \eta - \frac{1}{2} k g^{\mu\nu} (\nabla \eta)^2], \end{aligned} \quad (18a)$$

$$\Xi = \frac{1}{2\kappa} \sqrt{-g} [f' R_{\mathcal{M}} + p' + k' (\nabla^2 \eta) - 2 \nabla^\mu (k \nabla_\mu \eta)] \quad (18b)$$

are the momenta conjugated to the metric and to the dilaton on the manifold \mathcal{M} ,

$$\mathbf{P}^{\mu\nu} = -\frac{\sqrt{h}}{2\kappa} [f (K^{\mu\nu} - \text{tr } K h^{\mu\nu}) + h^{\mu\nu} \mathbf{L}_u f], \quad (19a)$$

$$\mathbf{P} = \frac{\sqrt{h}}{\kappa} (f' \text{tr } K - k \mathbf{L}_u \eta) \quad (19b)$$

are the momenta conjugated to $h_{\mu\nu}$ and $\eta|_{\mathcal{S}}$ on \mathcal{S} ,

$$\mathbf{\Pi}^{\mu\nu} = \frac{\varepsilon}{2\kappa} \sqrt{|\gamma|} [f (\Theta^{\mu\nu} - \text{tr } \Theta \gamma^{\mu\nu}) + \gamma^{\mu\nu} \mathbf{L}_{\tilde{n}} f], \quad (20a)$$

$$\mathbf{\Pi} = -\frac{\varepsilon}{\kappa} \sqrt{|\gamma|} (f' \text{tr } \Theta - k \mathbf{L}_{\tilde{n}} \eta) \quad (20b)$$

are the momenta conjugate to $\gamma_{\mu\nu}$ and $\eta|_{\mathcal{B}}$ on \mathcal{B} , and finally

$$\boldsymbol{\pi}^{\mu\nu} = \frac{1}{2\kappa} \sqrt{\sigma} f \alpha \sigma^{\mu\nu}, \quad (21a)$$

$$\boldsymbol{\pi} = \frac{1}{\kappa} \sqrt{\sigma} f' \alpha \quad (21b)$$

are the momenta conjugate to $\sigma_{\mu\nu}$ and $\eta|_{\mathcal{P}}$ on \mathcal{P} .

3 Derivation of the Hamiltonians

In this section we derive two Hamiltonians, corresponding to two different Legendre transformations of the action (16), which are the generalizations for a dilaton gravity theory of those proposed by Hawking and Hunter [1], and Booth and Mann [2].

First Hamiltonian

The action (16) is first decomposed with respect to the foliation in the standard way, using the Gauss-Codazzi equations

$$R_{\mathcal{M}} = R_{\mathcal{S}} + K^{\mu\nu} K_{\mu\nu} - (\text{tr } K)^2 - 2\nabla_{\mu}(u^{\mu} \text{tr } K + a^{\mu}), \quad (22)$$

and the decomposition of the squared divergence of the dilaton

$$(\nabla\eta)^2 = (\mathbf{D}\eta)^2 - (\mathbf{L}_u \eta)^2. \quad (23)$$

One obtains:

$$\begin{aligned} \mathfrak{J} = & \frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-g} \{ f R_{\mathcal{S}} + f K^{\mu\nu} K_{\mu\nu} - f (\text{tr } K)^2 + k (\mathbf{D}\eta)^2 - k (\mathbf{L}_u \eta)^2 \\ & + p + 2 \text{tr } K \mathbf{L}_u f + 2 \mathbf{L}_a f - 2 \nabla_{\mu} [f (u^{\mu} \text{tr } K + a^{\mu})] \} \\ & + \frac{1}{\kappa} \int_{S'}^{S''} \sqrt{h} f \text{tr } K - \frac{\varepsilon}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} f \text{tr } \Theta + \frac{1}{\kappa} \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\sigma} f \alpha. \end{aligned} \quad (24)$$

We can rewrite the intrinsic curvature $K^{\mu\nu}$ and the Lie derivative of the dilaton $\mathbf{L}_u \eta$ in terms of the momenta $\mathbf{P}^{\mu\nu}$ and \mathbf{P} :

$$K^{\mu\nu} = \frac{\kappa}{f Q \sqrt{h}} \{ -2Q \mathbf{P}^{\mu\nu} + 2[(f')^2 - f k] \text{tr } \mathbf{P} h^{\mu\nu} + f f' \mathbf{P}^{\mu\nu} \}, \quad (25a)$$

$$\mathbf{L}_u \eta = \frac{\kappa}{Q \sqrt{h}} [-2f' \text{tr } \mathbf{P} + (d-1) f \mathbf{P}], \quad (25b)$$

with

$$Q \stackrel{\text{def}}{=} d(f')^2 - (d-1) f k. \quad (26)$$

Using the previous equations, together with the relation

$$\mathbf{L}_a f = \frac{1}{N} \mathbf{D}_{\mu} (N \mathbf{D}^{\mu} f) - \mathbf{D}^2 f, \quad (27)$$

we get

$$\begin{aligned} \mathfrak{J} = & \int_{\mathcal{M}} \left[\mathbf{P}^{\mu\nu} \mathbf{L}_X h_{\mu\nu} + \mathbf{P}^{\mu\nu} \mathbf{L}_X \eta - N \mathbf{H}^{\perp} - V^{\mu} \mathbf{H}_{\mu} \right. \\ & - \frac{1}{\kappa} \sqrt{-g} \nabla_{\mu} (u^{\mu} f \text{tr } K + a^{\mu} f) \\ & \left. + \frac{1}{\kappa} \sqrt{h} \mathbf{D}_{\mu} \left(N \mathbf{D}^{\mu} f - 2V_{\nu} \frac{\kappa}{\sqrt{h}} \mathbf{P}^{\mu\nu} \right) \right] \\ & + \frac{1}{\kappa} \int_{S'}^{S''} \sqrt{h} f \text{tr } K - \frac{\varepsilon}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} f \text{tr } \Theta + \frac{1}{\kappa} \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\sigma} f \alpha, \end{aligned} \quad (28)$$

where the Hamiltonian constraints \mathbf{H}^\perp and \mathbf{H}_μ are given by

$$\begin{aligned} \mathbf{H}^\perp \stackrel{\text{def}}{=} & \frac{2\kappa}{\sqrt{h}} \left[\frac{1}{f} \mathbf{P}^{\mu\nu} \mathbf{P}_{\mu\nu} - \frac{(f')^2 - fk}{fQ} (\text{tr } \mathbf{P})^2 - \frac{f'}{Q} \text{tr } \mathbf{P} \mathbf{P} \right. \\ & \left. + \frac{(d-1)f}{4Q} (\mathbf{P}^\perp)^2 \right] - \frac{\sqrt{h}}{2\kappa} [fR_S + k(\mathbf{D}\eta)^2 + p - 2\mathbf{D}^2 f], \end{aligned} \quad (29a)$$

$$\mathbf{H}_\mu \stackrel{\text{def}}{=} -2\mathbf{D}_\nu \mathbf{P}^\nu_\mu + \mathbf{P}^\perp_\mu \mathbf{D}_\mu \eta, \quad (29b)$$

and Q is defined in Eq. (26).

Let us now focus on the additional boundary terms in Eq. (28) that represent total derivatives. The first term yields boundary terms on \mathcal{S}' , \mathcal{S}'' , and \mathcal{B} :

$$\begin{aligned} \mathfrak{I}_1 &= -\frac{1}{\kappa} \int_{\mathcal{M}} \sqrt{-g} \nabla_\mu (u^\mu f \text{tr } K + a^\mu f) \\ &= \frac{1}{\kappa} \int_{\mathcal{S}''} \sqrt{h} (-f \text{tr } K + u_\mu a^\mu f) - \frac{\varepsilon}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} (f\beta \text{tr } K + f\tilde{n}^\mu a_\mu). \end{aligned} \quad (30)$$

Since a^μ is orthogonal to u^μ , we see that the first integral in \mathfrak{I}_1 exactly cancels out with the \mathcal{S} -integral already present in the action (see Eq. (16)). The second integral, instead, sums up with the \mathcal{B} -integral of the action to give

$$\mathfrak{I}_2 = -\frac{\varepsilon}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} (f \text{tr } \Theta + f\beta \text{tr } K + f\tilde{n}_\mu a^\mu). \quad (31)$$

Using Eq. (10) one can show that Eq. (31) can be written as

$$\mathfrak{I}_2 = -\frac{1}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} (f\lambda^{-1} \text{tr } \theta + f\tilde{u}^\mu \Delta_\mu \alpha). \quad (32)$$

Taking out of it a total divergence by using $f\tilde{u}^\mu \Delta_\mu \alpha = \Delta_\mu (f\tilde{u}^\mu \alpha) - \alpha \Delta_\mu (f\tilde{u}^\mu)$, one finds

$$\mathfrak{I}_2 = -\frac{1}{\kappa} \int_{\mathcal{B}} \sqrt{|\gamma|} [f \text{tr } \theta - \alpha \Delta_\mu (f\tilde{u}^\mu)] - \frac{1}{\kappa} \int_{\mathcal{P}'} \sqrt{\sigma} f \alpha, \quad (33)$$

so that the \mathcal{P} -integral in \mathfrak{I}_2 exactly cancels out with the \mathcal{P} -integral which appears in the action (16). Let us now consider the last divergence in Eq. (28). This term yields the following boundary contribution:

$$\begin{aligned} \mathfrak{I}_3 &= \frac{1}{\kappa} \int_t \int_{\mathcal{S}_t} \sqrt{h} \mathbf{D}_\mu \left(N \mathbf{D}^\mu f - 2V_\nu \frac{\kappa}{\sqrt{h}} \mathbf{P}^{\mu\nu} \right) \\ &= \frac{1}{\kappa} \int_t \int_{\mathcal{P}_t} \sqrt{h} \left(N n^\mu \mathbf{D}_\mu f - 2n_\mu V_\nu \frac{\kappa}{\sqrt{h}} \mathbf{P}^{\mu\nu} \right). \end{aligned} \quad (34)$$

It follows that the action put into canonical form contains only a boundary integral on \mathcal{B} :

$$\begin{aligned} \mathfrak{I} = & \int_t \left\{ \int_{\mathcal{S}_t} (\mathbf{P}^{\mu\nu} \mathbf{L}_X h_{\mu\nu} + \mathbf{P} \cdot \mathbf{L}_X \eta - N \mathbf{H}^\perp - V^\mu \mathbf{H}_\mu) \right. \\ & \left. - \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{\sigma} \left[N[f \mathbf{tr} \theta - \lambda \alpha \Delta_\mu (f \tilde{u}^\mu) - n^\mu \mathbf{D}_\mu f] - 2V_\mu n_\nu \frac{\kappa}{\sqrt{h}} \mathbf{P}^{\mu\nu} \right] \right\}. \end{aligned} \quad (35)$$

It is now straightforward to perform the Legendre transformation,

$$\int_t \mathfrak{H} \equiv \int_t \int_{\mathcal{S}_t} (\mathbf{P}^{\mu\nu} \mathbf{L}_X h_{\mu\nu} + \mathbf{P} \cdot \mathbf{L}_X \eta) - \mathfrak{I}, \quad (36)$$

which gives us the Hamiltonian:

$$\mathfrak{H} = \int_{\mathcal{S}_t} (N \mathbf{H}^\perp + V^\mu \mathbf{H}_\mu) + \int_{\mathcal{P}_t} (N \mathbf{E} - V^\mu \mathbf{J}_\mu), \quad (37)$$

where

$$\mathbf{E} \stackrel{\text{def}}{=} \frac{\sqrt{\sigma}}{\kappa} [f \mathbf{tr} \theta - \mathbf{L}_n f - \lambda \alpha \Delta_\mu (f \tilde{u}^\mu)], \quad (38a)$$

$$\mathbf{J}_\mu \stackrel{\text{def}}{=} -2n_\nu \frac{\mathbf{P}^{\mu\nu}}{\sqrt{h}}. \quad (38b)$$

Notice that \mathbf{E} contains an (annoying) explicit dependence on the intersection angle α , just like it happens for the HH Hamiltonian. In fact, the $N \mathbf{E}$ integral reduces, in the non-dilatonic case, to the sum of the HH ‘curvature’ and ‘tilting’ terms, whereas the $V^\mu \mathbf{J}_\mu$ integral reduces to the HH ‘momentum’ term. In order to get rid of the explicit angle dependence we have to subtract a reference term from the Hamiltonian.

Second Hamiltonian

In this subsection we use Booth and Mann’s prescription, i.e. we require the time-flow vector field X^μ to lie on the outer boundary \mathcal{B} (so that $\tilde{n}_\mu X^\mu = 0$). This means that we are focusing our attention on the foliation of the outer boundary \mathcal{B} into surfaces \mathcal{P}_t , rather than on the foliation of the spacetime \mathcal{M} into surfaces \mathcal{S}_t . This, in turn, implies that we have to pass from the spacetime lapse N and shift V^μ to the *boundary* lapse \tilde{N} and shift \tilde{V}^μ , and from the quantities $\{\mathbf{tr} \theta, u^\mu, n^\mu\}$ to the quantities $\{\mathbf{tr} \tilde{\theta}, \tilde{u}^\mu, \tilde{n}^\mu\}$.

Let us now write the boundary contributions in the Hamiltonian (37) in terms of the new objects. Summing up the following identities, which are ob-

tained from Eqs. (11), (19a), (4), and (6):

$$N f \mathbf{tr} \theta = \varepsilon N \lambda f \mathbf{tr} \tilde{\theta} + \varepsilon N \lambda \beta f \mathbf{tr} K + \varepsilon N \lambda \beta f n^\mu (\tilde{n}^\nu \nabla_\mu \tilde{u}_\nu + \nabla_\mu \alpha), \quad (39)$$

$$2n_\mu V_\nu \frac{\kappa}{\sqrt{h}} P^{\mu\nu} = -\varepsilon N \lambda \beta f \mathbf{tr} K + \varepsilon N \lambda \beta \mathbf{L}_u f + f V^\mu (\tilde{n}^\nu \nabla_\mu \tilde{u}_\nu + \nabla_\mu \alpha), \quad (40)$$

$$-N \mathbf{L}_n f = -\varepsilon N \lambda \mathbf{L}_{\tilde{n}} f - \varepsilon N \lambda \beta \mathbf{L}_u f, \quad (41)$$

and using the relations $\tilde{N} = N \lambda$ and $\tilde{V}^\mu = \varepsilon N \lambda \beta n^\mu + V^\mu$, we find that the boundary integral in Eq. (37) can be expressed as follows:

$$\begin{aligned} \int_{\mathcal{P}_t} (N \mathbf{E} - V^\mu \mathbf{J}_\mu) = & \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{\sigma} [\varepsilon \tilde{N} (f \mathbf{tr} \tilde{\theta} - \mathbf{L}_{\tilde{n}} f) + f \tilde{V}^\mu \tilde{n}^\nu \nabla_\mu \tilde{u}_\nu \\ & + f \tilde{V}^\mu \Delta_\mu \alpha - \tilde{N} \alpha \Delta_\mu (f \tilde{u}^\mu)]. \end{aligned} \quad (42)$$

Let us now consider the terms containing α , which can be manipulated using the identities $\sqrt{|\gamma|} = \tilde{N} \sqrt{\sigma}$ and $X^\mu = \tilde{N} \tilde{u}^\mu + \tilde{V}^\mu$, to obtain:

$$\begin{aligned} \int_{\mathcal{P}_t} \sqrt{\sigma} [f \tilde{V}^\mu \Delta_\mu \alpha - \tilde{N} \alpha \Delta_\mu (f \tilde{u}^\mu)] = \\ = \int_{\mathcal{P}_t} \sqrt{|\gamma|} \left[\Delta_\mu \left(f \frac{\tilde{V}^\mu}{\tilde{N}} \alpha \right) - \alpha \Delta_\mu \left(f \frac{\tilde{N} \tilde{u}^\mu + \tilde{V}^\mu}{\tilde{N}} \right) \right] \\ = \int_{\mathcal{P}_t} \sqrt{|\gamma|} \Delta_\mu \left(f \frac{\tilde{V}^\mu}{\tilde{N}} \alpha \right) - \int_{\mathcal{P}_t} \alpha \mathbf{L}_X (f \sqrt{\sigma}). \end{aligned} \quad (43)$$

Note that the integral containing the total divergence can be discarded, since, by Stokes' theorem, upon integration in time it gives vanishing terms proportional to $\tilde{u}_\mu \tilde{V}^\mu = 0$ on \mathcal{P}' and \mathcal{P}'' . Moreover, it is easy to show that

$$\alpha \mathbf{L}_X (f \sqrt{\sigma}) \equiv \pi^{\mu\nu} \mathbf{L}_X \sigma_{\mu\nu} + \pi \mathbf{L}_X \eta. \quad (44)$$

Using the previous equations we finally find

$$\begin{aligned} \int_{\mathcal{P}_t} (N \mathbf{E} - V^\mu \mathbf{J}_\mu) = & \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{\sigma} [\varepsilon \tilde{N} (f \mathbf{tr} \tilde{\theta} - \mathbf{L}_{\tilde{n}} f) - \tilde{V}^\mu f \tilde{u}^\nu \nabla_\mu \tilde{n}_\nu] \\ & - \int_{\mathcal{P}_t} (\pi^{\mu\nu} \mathbf{L}_X \sigma_{\mu\nu} + \pi \mathbf{L}_X \eta). \end{aligned} \quad (45)$$

The last term can be discarded if we perform a Legendre transformation different from (36):

$$\int_t \tilde{\mathfrak{H}} \equiv \int_t \left[\int_{\mathcal{S}_t} (\mathbf{P}^{\mu\nu} \mathbf{L}_X h_{\mu\nu} + \mathbf{P} \mathbf{L}_X \eta) + \int_{\mathcal{P}_t} (\pi^{\mu\nu} \mathbf{L}_X \sigma_{\mu\nu} + \pi \mathbf{L}_X \eta) \right] - \mathfrak{I}. \quad (46)$$

In this way $\sigma_{\mu\nu}$, $\eta|_{\mathcal{P}}$, and $\pi^{\mu\nu}$, π are treated as canonical variables and momenta on the same footing as $h_{\mu\nu}$, $\eta|_S$, and $P^{\mu\nu}$, P . We now have the new Hamiltonian:

$$\tilde{\mathfrak{H}} = \int_{\mathcal{S}_t} (N \mathbf{H}^\perp + V^\mu \mathbf{H}_\mu) + \int_{\mathcal{P}_t} (\tilde{N} \tilde{\mathbf{E}} - \tilde{V}^\mu \tilde{\mathbf{J}}_\mu), \quad (47)$$

where

$$\tilde{\mathbf{E}} \stackrel{\text{def}}{=} \frac{\varepsilon}{\kappa} \sqrt{\sigma} (f \operatorname{tr} \tilde{\theta} - \mathbf{L}_{\tilde{n}} f), \quad (48a)$$

$$\tilde{\mathbf{J}}_\mu \stackrel{\text{def}}{=} - \frac{\sqrt{\sigma}}{\kappa} f \tilde{u}^\nu \nabla_\mu \tilde{n}_\nu. \quad (48b)$$

This Hamiltonian reduces, in the non-dilatonic case and when \mathcal{B} is timelike, to Booth and Mann's Hamiltonian. Analogously to the BM Hamiltonian, $\tilde{\mathfrak{H}}$ has no explicit dependence upon the intersection angle between \mathcal{B} and \mathcal{S}_t . This happens because all quantities in the boundary term of (47) are defined considering a local, natural spacetime foliation of \mathcal{M} into slices $\tilde{\mathcal{S}}_t$ orthogonal to the outer boundary.

Background terms

It is a well-known fact that we can subtract from the gravitational action (and thus from the Hamiltonian) a reference term \mathfrak{J} , which has to be a functional of the boundary metric only, without affecting the equations of motion of the system. Subtracting such a term corresponds to redefining the zero-point energy and momentum [16]. This may be necessary when we want to renormalize divergent quantities, which may appear in the Hamiltonian when we consider an outer boundary at infinity. Usually one chooses this term in order to have vanishing energy and momentum for a given reference spacetime (e.g. Minkowski or anti-de Sitter spacetime).

For the Hamiltonian \mathfrak{H} of Eq. (37), the reference term can be defined in the following way (note that another equivalent, but local, definition can be given along the lines of [2, Sect. III D]). First we embed the boundary $(\mathcal{B}, \gamma_{\mu\nu}, \eta|_{\mathcal{B}})$ into the reference spacetime $(\underline{\mathcal{M}}, g_{\mu\nu}, \eta)$ in such a way that the metric and dilaton induced on \mathcal{B} by the embedding agree with those induced from \mathcal{M} (we may call it an isometric and 'isodilatonic' embedding). Moreover we must require that $\underline{\mathcal{M}}$ be foliated in such a way that β (see Eq. (2)) has the same value in \mathcal{M} and in $\underline{\mathcal{M}}$.

These conditions together imply that $\sqrt{\sigma}N$, λ , α , Δ_μ , \tilde{u}^μ , and $f|_{\mathcal{B}}$ are the same in the two spacetimes. Hence, we define the reference term to be

$$\tilde{\mathfrak{H}}_{\mathcal{P}_t} \stackrel{\text{def}}{=} - \int_{\mathcal{P}_t} (N \mathbf{E} - V^\mu \mathbf{J}_\mu) \quad \text{calculated with respect to } \underline{\mathcal{M}}. \quad (49)$$

With this definition the boundary term $\mathfrak{H}_{\mathcal{P}_t}$ becomes explicitly:

$$\begin{aligned} \mathfrak{H}_{\mathcal{P}_t} = & \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{\sigma} \left\{ N[f(\mathbf{tr} \theta - \mathbf{tr} \underline{\theta}) - (\mathbf{L}_n f - \mathbf{L}_{\underline{n}} \underline{f})] \right. \\ & \left. - \kappa \left(V_\mu n_\nu \frac{P^{\mu\nu}}{\sqrt{h}} - \underline{V}_\mu \underline{n}_\nu \frac{\underline{P}^{\mu\nu}}{\sqrt{\underline{h}}} \right) \right\}, \end{aligned} \quad (50)$$

where all objects with an under-bar are evolved on the reference spacetime \mathcal{M} . Note that the term containing the explicit dependence on the hyperbolic angle α has disappeared, for it is the same in both spacetimes: this makes the presence of the reference term a necessary feature in the case of the Hamiltonian \mathfrak{H} of Eq. (37).

The situation is different in the case of the Hamiltonian $\tilde{\mathfrak{H}}$ of Eq. (47). We still require an isometric and isodilatational embedding of \mathcal{B} in the reference spacetime \mathcal{M} , but now we do not impose any requirement about the foliation of \mathcal{M} and the intersection angle; yet this weaker condition implies that the *boundary* lapse and shift agree in both spacetimes. The reference term is then defined by

$$\begin{aligned} \tilde{\mathfrak{H}}_{\mathcal{P}_t} & \stackrel{\text{def}}{=} - \int_{\mathcal{P}_t} (\tilde{N} \tilde{\underline{E}} - \tilde{V}^\mu \tilde{\underline{J}}_\mu) \quad \text{calculated with respect to } \mathcal{M} \\ & \equiv - \int_{\mathcal{P}_t} (\tilde{N} \tilde{\underline{E}} - \tilde{V}^\mu \tilde{\underline{J}}_\mu), \end{aligned} \quad (51)$$

where

$$\tilde{\underline{E}} \stackrel{\text{def}}{=} \frac{\varepsilon}{\kappa} \sqrt{\sigma} (f \mathbf{tr} \tilde{\underline{\theta}} - \mathbf{L}_{\tilde{\underline{n}}} f), \quad (52a)$$

$$\tilde{\underline{J}}_\mu \stackrel{\text{def}}{=} - \frac{1}{\kappa} \sqrt{\sigma} f \tilde{u}^\nu \nabla_\mu \tilde{n}_\nu; \quad (52b)$$

The boundary term in $\tilde{\mathfrak{H}}$ becomes now

$$\begin{aligned} \tilde{\mathfrak{H}}_{\mathcal{P}_t} & = \int_{\mathcal{P}_t} [\tilde{N}(\tilde{\underline{E}} - \underline{\underline{E}}) - \tilde{V}^\mu(\tilde{\underline{J}}_\mu - \underline{\underline{J}}_\mu)] \\ & = \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{\sigma} \{ \varepsilon \tilde{N} [f(\mathbf{tr} \tilde{\underline{\theta}} - \mathbf{tr} \underline{\underline{\theta}}) - (\mathbf{L}_{\tilde{\underline{n}}} f - \mathbf{L}_{\underline{\underline{n}}} \underline{\underline{f}})] \\ & \quad + f \tilde{V}^\mu \tilde{u}^\nu (\nabla_\mu \tilde{n}_\nu - \underline{\nabla}_\mu \underline{\underline{n}}_\nu) \}. \end{aligned} \quad (53)$$

In this case the reference term is not necessary to eliminate explicit angle dependence in the Hamiltonian, for $\tilde{\mathfrak{H}}$ has none by construction.

Null outer boundaries

All the results obtained so far hold, generally, for a spacetime with a timelike or spacelike outer boundary. The formalism developed in the previous subsections can deal with these two cases but is not meant to deal with a null outer boundary. The main reason is that the action (16) becomes ill-behaved whenever one

considers the limit of a null \mathcal{B} , for the integrand in the \mathcal{P} -surface integral diverge. One should define a new action with appropriate boundary terms before going on to derive Hamiltonians.

Yet, no one prevents us from considering a null \mathcal{B} as a limit of a parameter-depending timelike or spacelike boundary. \mathcal{B} becomes null when the parameter is sent, say, to zero.

This procedure is different for the two Hamiltonians \mathfrak{H} and $\tilde{\mathfrak{H}}$. In case of the Hamiltonian \mathfrak{H} there are no restrictions on the nature of the evolution generator $X^\mu = Nu^\mu + V^\mu$. We can have timelike, spacelike or null generators. The only problem here is that the Hamiltonian \mathfrak{H} has a term depending on the hyperbolic angle α , which blows up as we approach a null- \mathcal{B} limit. This divergence can be easily cured: it disappears upon subtracting a reference term as we have discussed in the previous subsection.

The case of Hamiltonian $\tilde{\mathfrak{H}}$ of Eq. (47) is more involved. In this case the evolution generator is required to lie on the outer boundary, so that there are some kinds of generators that we can study only if we take a null \mathcal{B} . Since this assumption cannot be imposed from the beginning, we are forced to resort to a limit procedure in order to study the case of null generators.

A quick analysis of the boundary integral present in the definition of $\tilde{\mathfrak{H}}$, given in Eq. (47), shows that $\tilde{\mathfrak{H}}$ has a finite limit when \mathcal{B} tends to a null hypersurface (both from inside and from outside). This can be easily shown by writing the divergent integral in a coordinate system where $\mathcal{B} = \{r = \text{const.}\}$ and $\tilde{\mathcal{S}}_t = \{t = \text{const.}\}$,

$$\begin{aligned} \int_{\mathcal{P}_t} (\tilde{N}\tilde{\mathbf{E}} - \tilde{V}^\mu \tilde{\mathbf{J}}_\mu) = \\ = \frac{1}{\kappa} \int_{\mathcal{P}_t} \sqrt{-g} \left(f \sigma^{\mu\nu} \Gamma^r_{\mu\nu} - \nabla^r f + f V^\mu g^{t\nu} \Gamma^r_{\mu\nu} - f \frac{\sqrt{h}}{\sqrt{\sigma}} V^r n^\mu g^{t\nu} \Gamma^r_{\mu\nu} \right), \end{aligned} \quad (54)$$

and noting that this expression contains only objects $(f, g_{\mu\nu}, V^\mu, h_{\mu\nu}, n^\mu, \sigma_{\mu\nu})$ which by construction or hypothesis are well-behaved.

4 Discussion

In this paper we have derived two Hamiltonians for a general $(d+1)$ -dimensional dilaton gravity theory, \mathfrak{H} (Eq. (37)) and $\tilde{\mathfrak{H}}$ (Eq. (37)), which generalize Hawking and Hunter's [1] and Booth and Mann's [2] Hamiltonian respectively. For the purposes of the present discussion we can call \mathfrak{H} 'bulk-oriented Hamiltonian' and $\tilde{\mathfrak{H}}$ 'boundary-oriented Hamiltonian'.

When we use the bulk-oriented Hamiltonian, we focus our attention mainly on the foliation of \mathcal{M} into spacelike hypersurfaces and we use the 'bulk' lapse N and shift V^μ , together with other \mathcal{S}_t -related objects ($\text{tr } \theta$, $\mathbf{L}_u \eta$, etc.). Conversely, when we use the boundary-oriented Hamiltonian we assume that the

initial surface \mathcal{P}' is time-evolved along \mathcal{B} , we restrict our attention to the boundary foliation and we use the boundary lapse \tilde{N} and shift \tilde{V}^μ , and other boundary objects ($\text{tr } \tilde{\theta}$, $\mathbf{L}_{\tilde{n}} \eta$, etc.).²

We have seen also that one Hamiltonian has merits where the other has drawbacks, and vice versa. The bulk-oriented one allows to consider all kinds of generators (spacelike, timelike, null), but contains an explicit dependence upon the hyperbolic angle of the foliation α —which diverges in the limit of a null outer boundary—needing an additive spacetime reference term. Conversely, the boundary-oriented Hamiltonian has no explicit dependence on the intersection between the slices and the outer boundary, yet forces us to modify the latter (considering even the possibility of a spacelike case), and in some cases to resort to limit procedures, in order to study a generic evolution generator.

Apart from the fact that the two Hamiltonians correspond to a different choice of thermodynamical ensembles (see e.g. Kijowski [13]), it is evident that they are useful in two complementary situations. \mathfrak{H} is the natural choice for the Hamiltonian when one is dealing with a spatially non-compact spacetime, whereas $\tilde{\mathfrak{H}}$ is useful in the case of a bounded spacetime.

In a spatially non-compact spacetime we may want to consider e.g. spatial translations, which usually belong to the group of automorphisms of the manifold. In this case the bulk-oriented Hamiltonian allows us to study the generators of translations. We first introduce a boundary at finite distance, study the generators on this boundary, then push the boundary to infinity to study the asymptotic behavior of our generators. The fact that the generators map the manifold out of the boundary is of no importance, since the boundary is introduced only to be pushed to infinity. Moreover, in this case the explicit dependence upon the angle foliation is not problematic: a spatially non-compact spacetime usually needs a reference spacetime for renormalizing possible divergences, and the dependence on the angle α can be eliminated together with the divergences by subtracting from \mathfrak{H} a reference term.

On the other hand, when we deal with a spatially bounded manifold, usually, we do not consider transformations like e.g. spatial translations, for they are not automorphisms of the spacetime. In this case one naturally uses the boundary-oriented Hamiltonian $\tilde{\mathfrak{H}}$, which has no dependence on α . Moreover, the formalism we have developed in this paper enables one to use $\tilde{\mathfrak{H}}$ in spacetimes with all kind of boundaries, hence makes $\tilde{\mathfrak{H}}$ as much versatile as \mathfrak{H} for the study of all kinds of generators.

²The bulk-oriented approach is used e.g. by Lau [8, 9], Hawking and Horowitz [14], Hawking and Hunter [1], and implicitly also by DeWitt [15], Regge and Teitelboim [4], Brown and Henneaux [5], Cadoni and Mignemi [6, 7], et al. The boundary-oriented approach is followed by Brown and York [16], Bose and Dadhich [17], Kijowski [13], Brown, Creighton and Mann [18], Creighton and Mann [3], Booth and Mann [2, 19], et al.

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